Excited States of the Nucleon in 2+1 flavour QCD

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CSSM Lattice Collaboration

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Outline

- Two point correlation function:

$$G_{ij}(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} \langle \Omega | T\{\chi_i(x)\bar{\chi}_j(0)\} | \Omega \rangle .$$

- Inserting completeness

$$\sum_{B,\vec{p'},s} |B,\vec{p'},s\rangle\langle B,\vec{p'},s| = I$$

- Then

$$G_{ij}(t,\vec{p}) = \sum_{B^+} \lambda_{B^+}\bar{\lambda}_{B^+} e^{-E_{B^+}t} \frac{\gamma.p_{B^+} + M_{B^+}}{2E_{B^+}} + \sum_{B^-} \lambda_{B^-}\bar{\lambda}_{B^-} e^{-E_{B^-}t} \frac{\gamma.p_{B^-} - M_{B^-}}{2E_{B^-}}$$

- At $\vec{p} = 0$

$$G_{ij}^{\pm}(t,\vec{0}) = \mathrm{Tr}_{\mathrm{sp}}[\Gamma_{\pm} G_{ij}(t,\vec{0})]$$
$$= \sum_{B^{\pm}} \lambda_i^{\pm} \bar{\lambda}_j^{\pm} e^{-M_{B^{\pm}} t}.$$

- Parity projection operator,

$$\Gamma_{\pm} = \frac{1}{2}(1 \pm \gamma_0).$$

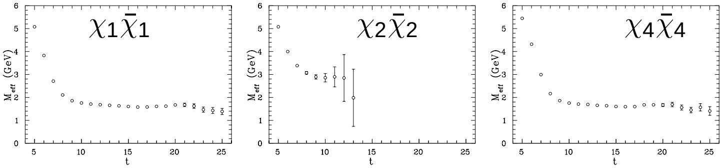
- Asymptotically

$$G_{ij}^{\pm}(t,\vec{0}) \overset{t \to \infty}{=} \lambda_{i0}^{\pm} \bar{\lambda}_{j0}^{\pm} e^{-M_{0^{\pm}} t}.$$

Interpolators

- Consider

$$\chi_1(x) = \epsilon^{abc}(u^{Ta}(x)\, C\gamma_5\, d^b(x))\, u^c(x)\,,$$
$$\chi_2(x) = \epsilon^{abc}(u^{Ta}(x)\, C\, d^b(x))\, \gamma_5\, u^c(x)\,,$$
$$\chi_4(x) = \epsilon^{abc}(u^{Ta}(x)\, C\gamma_5\gamma_4\, d^b(x))\, u^c(x).$$



Variational Method

- Consider N interpolating fields, then

$$\bar{\phi}^{\alpha} = \sum_{i=1}^{N} u_i^{\alpha} \bar{\chi}_i ,$$

$$\phi^{\alpha} = \sum_{i=1}^{N} v_i^{\alpha} \chi_i ,$$

such that,

$$\langle B_{\beta}, p, s | \bar{\phi}^{\alpha} | \Omega \rangle = \delta_{\alpha\beta} \bar{z}^{\alpha} \bar{u}(\alpha, p, s),$$

$$\langle \Omega | \phi^{\alpha} | B_{\beta}, p, s \rangle = \delta_{\alpha\beta} z^{\alpha} u(\alpha, p, s),$$

- Then a two point correlation function matrix for $\vec{p} = 0$, right multiplied by u_j^α has the property

$$G_{ij}^{\pm}(t)\, u_j^\alpha = \Big(\sum_{\vec{x}} \mathrm{Tr}_{\mathrm{sp}}\{\Gamma_\pm \langle\Omega|\chi_i\bar{\chi}_j|\Omega\rangle\}\Big)\, u_j^\alpha$$
$$= \lambda_i^\alpha \bar{z}^\alpha e^{-m_\alpha t}.$$

(no sum over α)

- The t dependence is contained in the exponential term

- This provides a recurrence relation at time $(t_0 + \triangle t)$,

$$G_{ij}(t_0 + \triangle t)u_j^\alpha = e^{-m_\alpha \triangle t} G_{ij}(t_0)u_j^\alpha .$$

- Multiplying by $[G_{ij}(t_0)]^{-1}$ from left,

$$[(G(t_0))^{-1} G(t_0 + \triangle t)]_{ij}\, u_j^\alpha = c^\alpha\, u_i^\alpha ,$$

- where $c^\alpha = e^{-m_\alpha \triangle t}$ is the eigenvalue.
- Similarly, it can also be solved for the left eigenvalue equation for v^α eigenvector,

$$v_i^\alpha\, [G(t_0 + \triangle t)(G(t_0))^{-1}]_{ij} = c^\alpha\, v_j^\alpha .$$

- The vectors u_j^α and v_i^α diagonalize the correlation matrix at time t_0 and $t_0 + \triangle t$ making the projected correlation function

$$v_i^\alpha G_{ij}(t) u_j^\beta = \delta^{\alpha\beta} z^\alpha \bar{z}^\beta e^{-m_\alpha t}.$$

- The projected correlator, is then analyzed to obtain masses of different states,

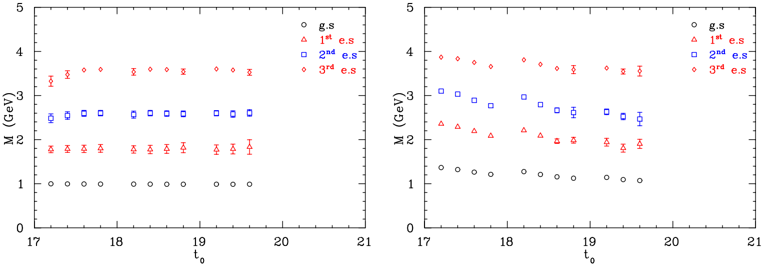
$$v_i^\alpha G_{ij}^{\pm}(t) u_j^\alpha \equiv G_{\pm}^\alpha,$$

- Our effective mass is defined as

$$M_{\rm eff}^\alpha(t) = \ln\left(\frac{G_{\pm}^\alpha(t,\vec{0})}{G_{\pm}^\alpha(t+1,\vec{0})}\right).$$

4×4 correlation matrix of χ_1 with 4 smearing levels

Projected Mass Vs Mass From Eigenvalue



- $t_{\text{start}} = t_0$ is shown in major tick marks
- $\triangle t$ is shown in minor tick marks

PACS-CS Simulation Details

PACS-CS Collaboration: S. Aoki, et al., Phys. Rev. **D79** (2009) 034503.

- Lattice volume: $32^3 \times 64$
- Non-perturbative $\mathcal{O}(a)$-improved Wilson quark action
- Iwasaki gauge action
- $2 + 1$ flavour dynamical-fermion QCD
- $\beta = 1.9$ providing $a = 0.0907$ fm
- $K_{ud} = \{$ 0.13700, 0.13727, 0.13754, 0.13770, 0.13781 $\}$
- $K_s = 0.13640$
- Lightest pion mass is 156 MeV.

Sommer Scale

- Lattice spacing is set via the force between static quarks

$$r_c^2 \left. \frac{\partial V(r)}{\partial r} \right|_{r=r_c} = c$$

- Sommer prefers $c = 1.65$, such that $r_c = r_0 = 0.49$ fm
- The Sommer scale facilitates comparisons with other results

Source Smearing

Correlation matrices are built from a variety of source and sink smearings.

$$\psi_i(x,t) = \sum_{x'} F(x,x')\psi_{i-1}(x',t),$$

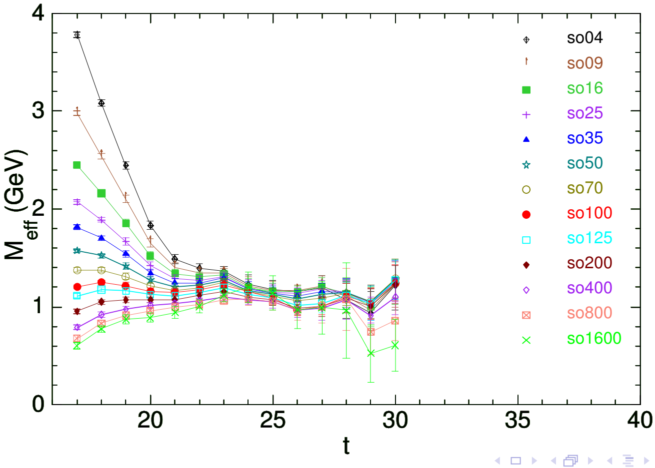
where,

$$\begin{aligned} F(x,x') &= (1-\alpha)\delta_{x,x'} + \frac{\alpha}{6}\sum_{\mu=1}^{3}[U_\mu(x)\delta_{x',x+\hat{\mu}} \\ &\quad + U_\mu^\dagger(x-\hat{\mu})\delta_{x',x-\hat{\mu}}], \end{aligned}$$

Fixing $\alpha = 0.7$, the procedure is repeated $N_{\rm sm}$ times.

Smeared Source - Point Sink Effective Masses

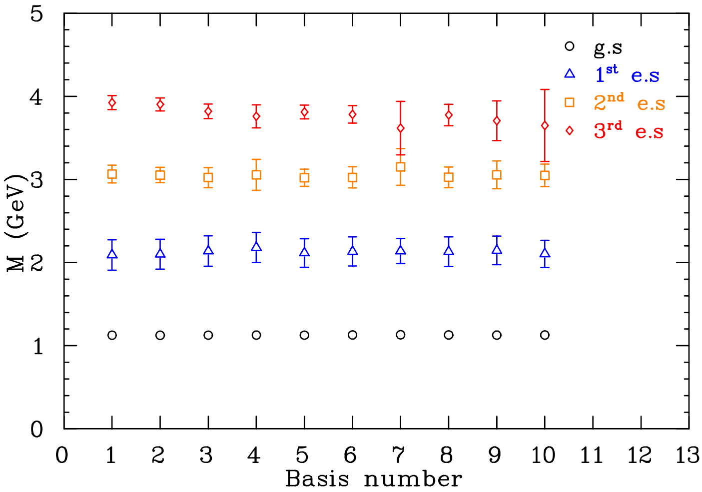
For second lightest quark mass and 50 configurations



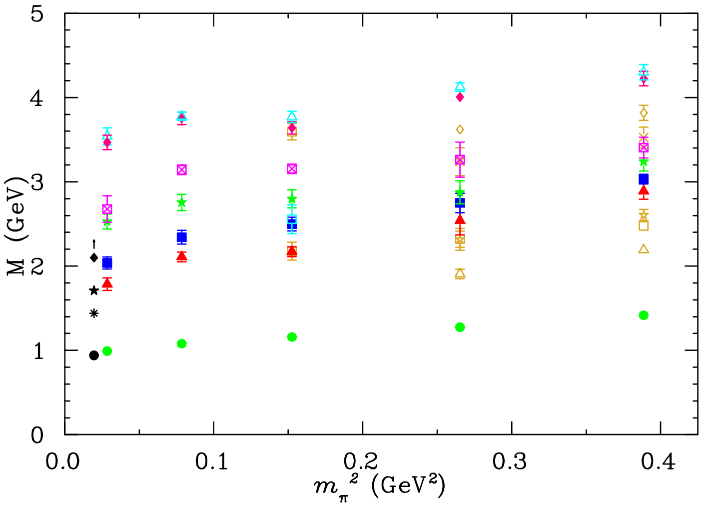
4×4 bases of $\chi_1\bar{\chi}_1$

Sweeps $\rightarrow$	16	25	35	50	70	100	125	200	400	800
Basis No. $\downarrow$						Bases				
1	16	-	35	-	70	100	-	-	-	-
2	16	-	35	-	70	-	125	-	-	-
3	16	-	35	-	-	100	-	200	-	-
4	16	-	35	-	-	100	-	-	400	-
5	16	-	-	50	-	100	125	-	-	-
6	16	-	-	50	-	100	-	200	-	-
7	16	-	-	50	-	-	125	-	-	800
8	-	25	-	50	-	100	-	200	-	-
9	-	25	-	50	-	100	-	-	400	-
10	-	-	35	-	70	-	125	-	400	-

All 4×4 bases: second lightest mass



Even Parity Nucleon Spectrum in full QCD



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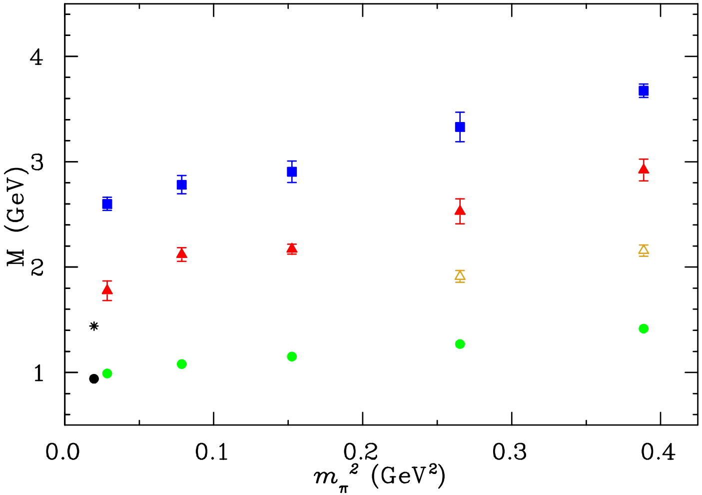
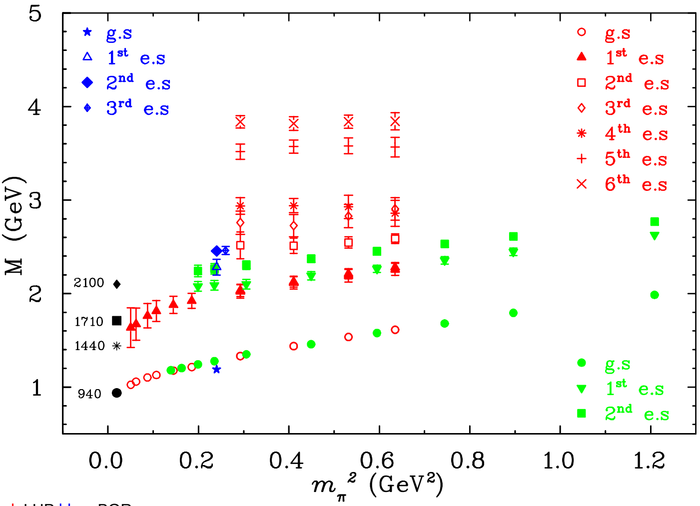


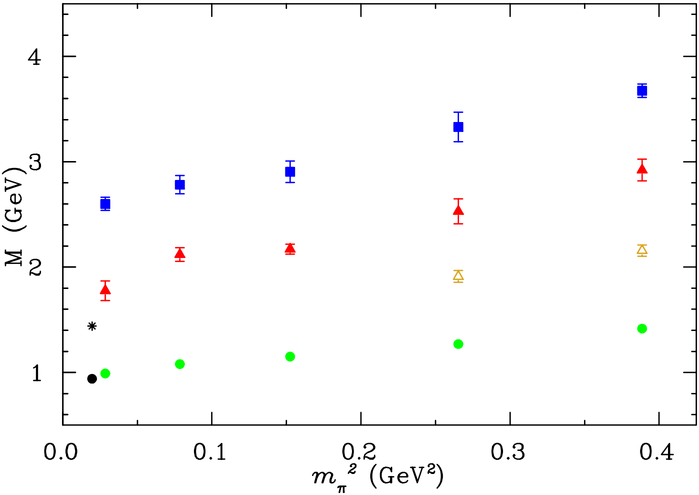
Configurations: Lightest = 750, rest = 350

Even Parity Nucleon Spectrum in Quenched QCD



CSSM red, LHP blue, BGR green.

Even Parity Nucleon Spectrum in full QCD



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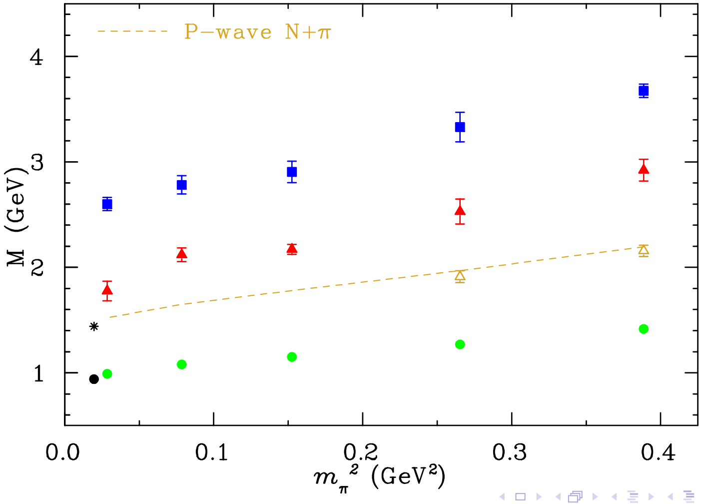
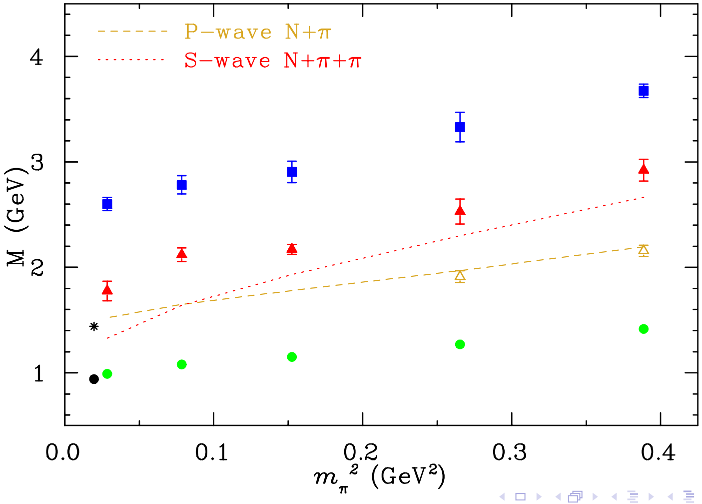
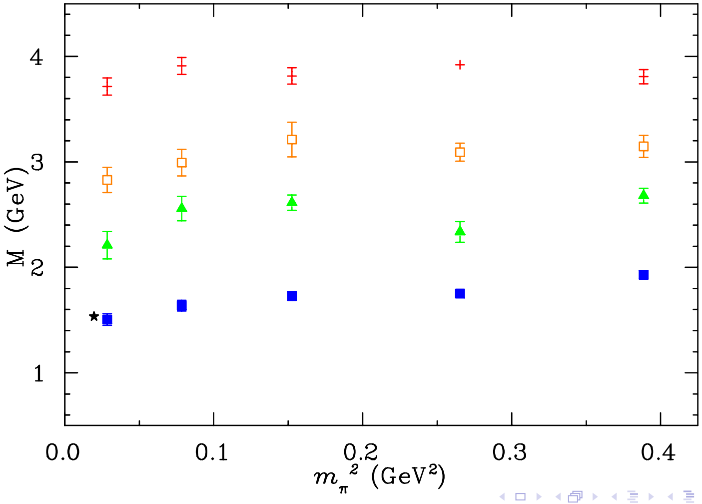


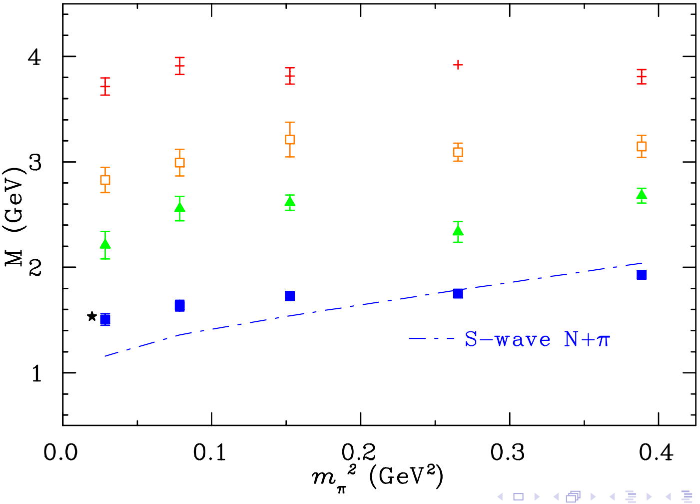
Even Parity Nucleon Spectrum in full QCD



$N\frac{1}{2}^-$ (1535) State



$N\frac{1}{2}^-$ (1535) State



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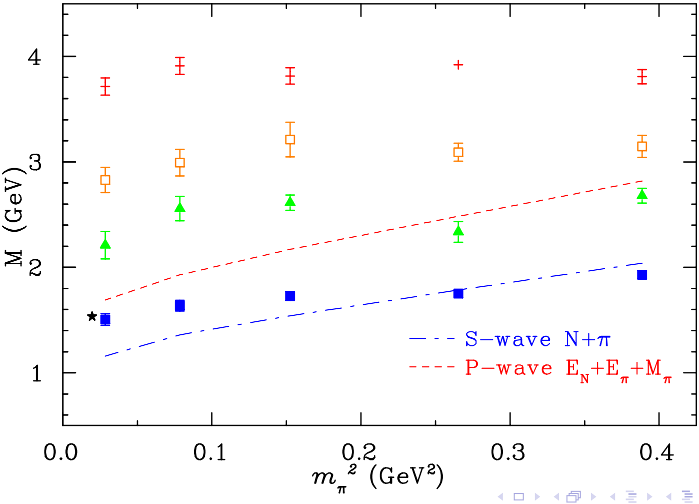
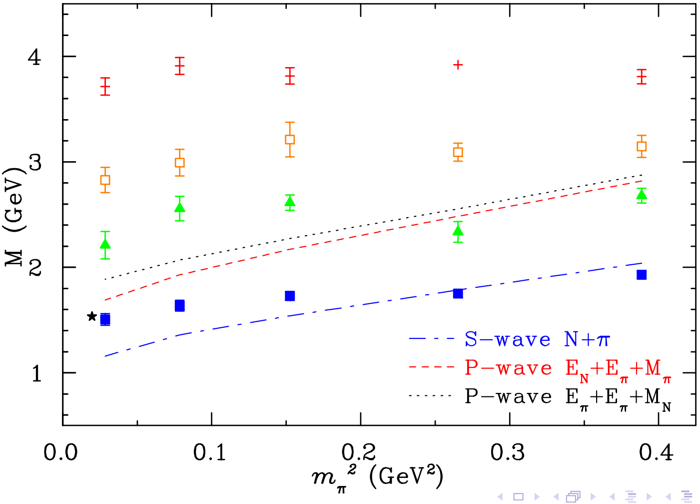


$N\frac{1}{2}^-$ (1535) State

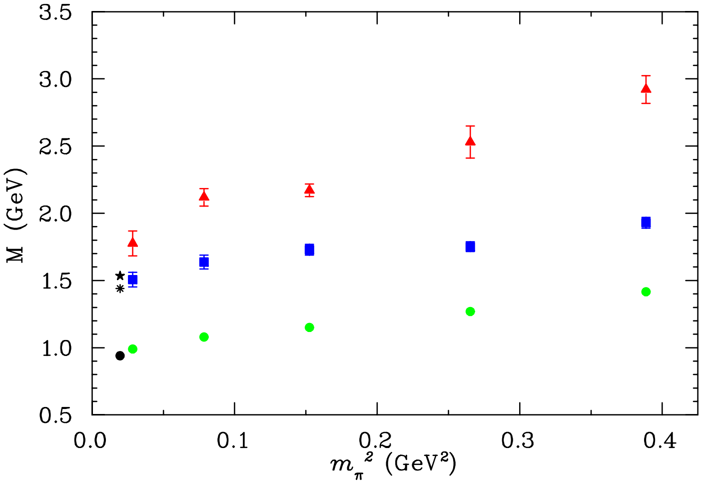


Loss of multi-particle states at light quark masses

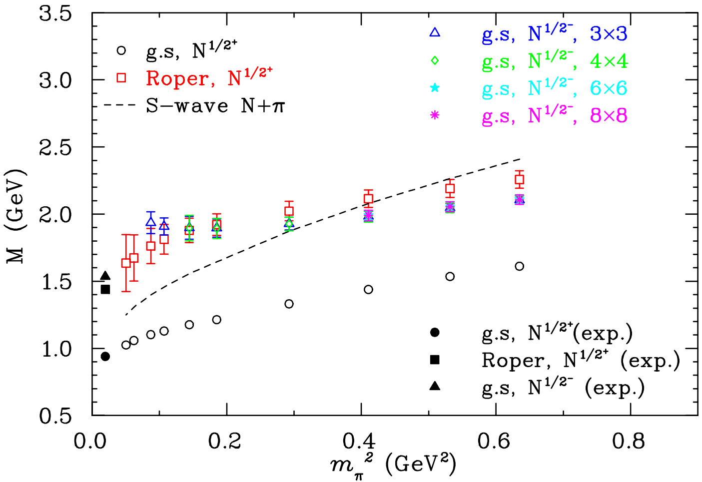
Consider an interpolator creating a resonance with a finite width in the infinite volume limit.
On a finite volume lattice:

- The energy thresholds of the states involve discretised momenta $n\,(2\,\pi/L)$.
- The density of states increases with the lattice volume $V = L^3$.
- The coupling to the meson-baryon states is suppressed by $1/\sqrt{V}$.
- This provides a finite width and finite spectral strength as $V \to \infty$.
- The strength of the volume suppression effect depends on the resonance width.
- The width of resonances with heavy quarks are relatively small.

$N\frac{1}{2}^+(940)$, $N\frac{1}{2}^+(1440)$, $N\frac{1}{2}^-(1535)$ States

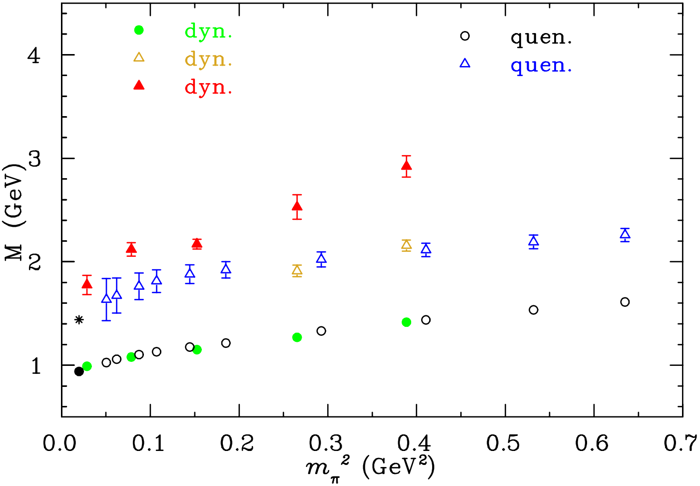


Roper and $N1/2^-$ states in Quenched QCD



Mahbub *et al.*, Phys. Lett. B **693**, 351 (2010), [arXiv:hep-lat/1007.4871].

Quenched Vs Dynamical, N^+ states



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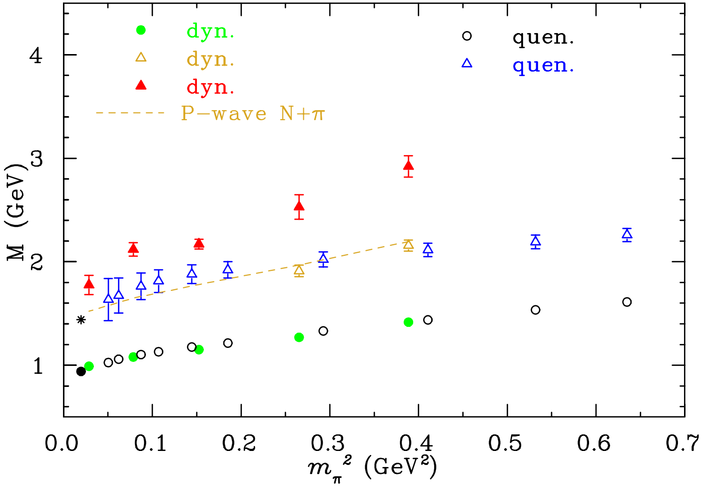
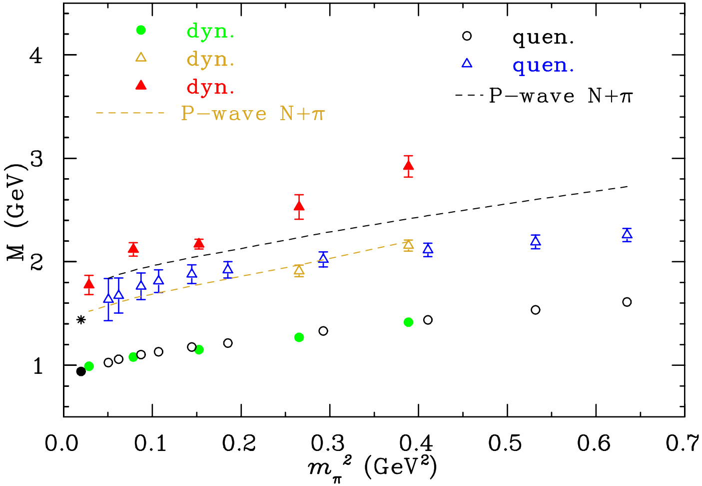
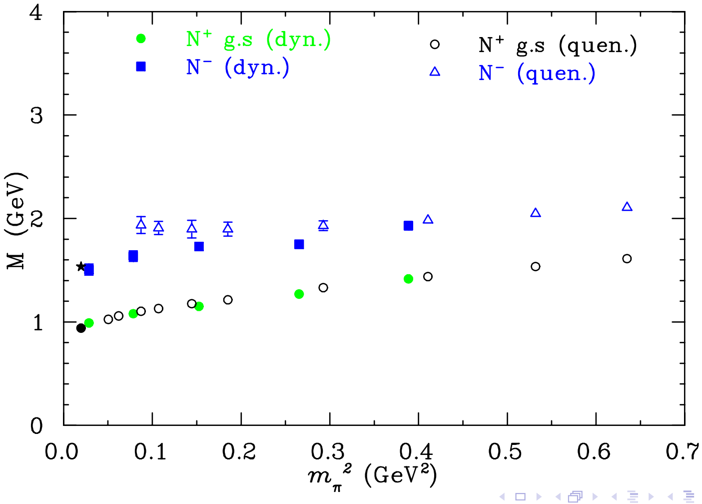


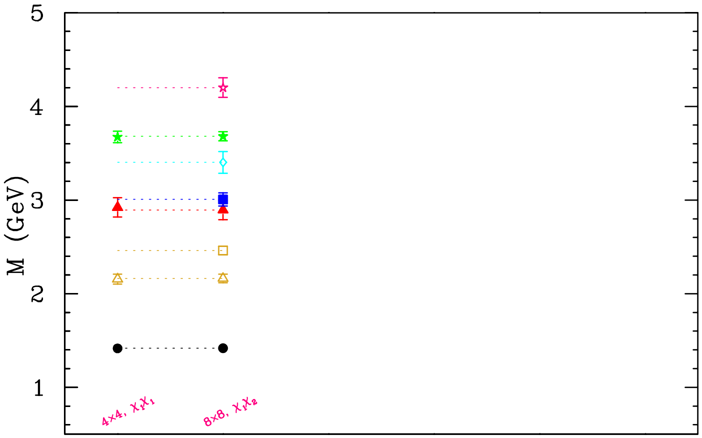
Quenched Vs Dynamical, N^+ states



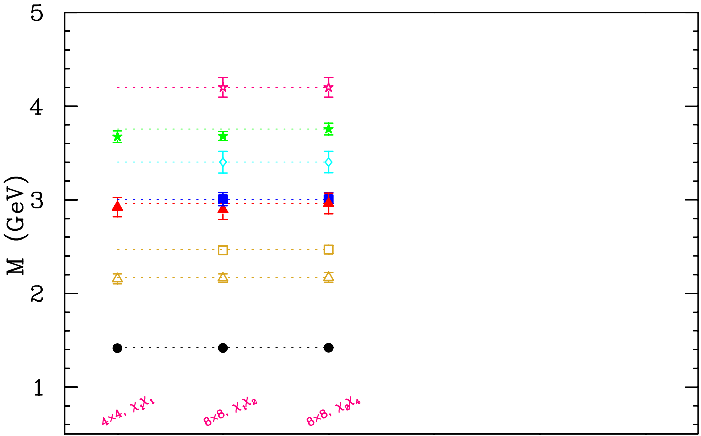
Quenched Vs Dynamical, N^- states



N^+ Spectrum for heaviest m_q: $4 \times 4 \rightarrow 8 \times 8$ $\chi_1 \chi_2$



N^+ Spectrum for heaviest m_q: 8×8 $\chi_2\,\chi_4$



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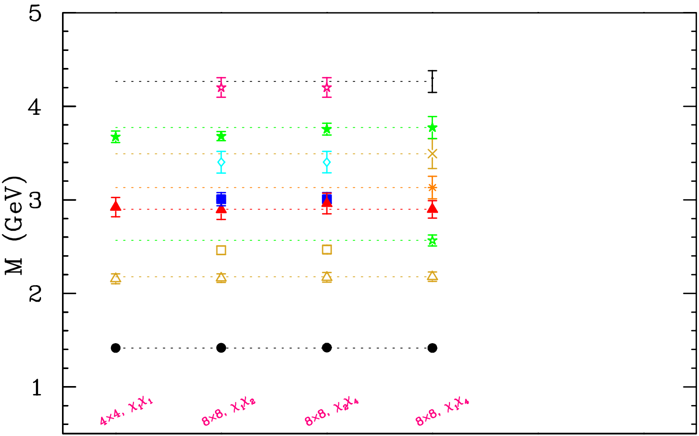
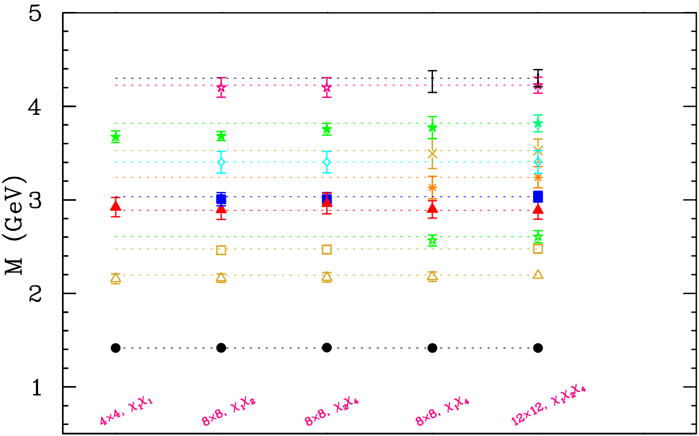
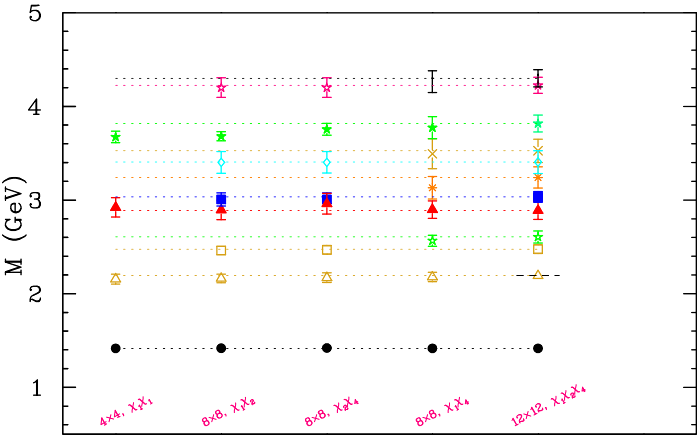


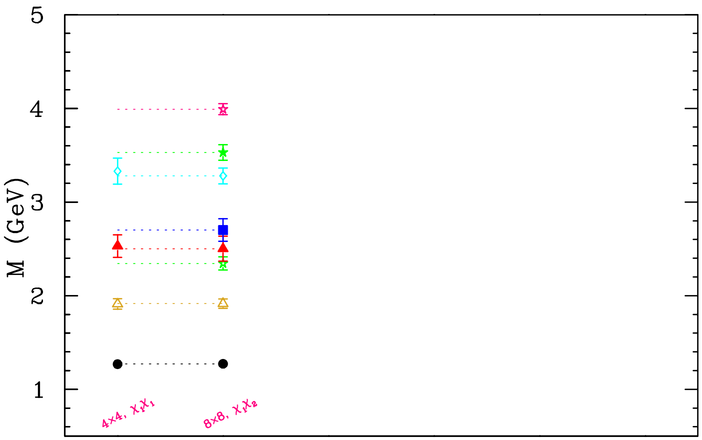
N^+ Spectrum for heaviest m_q: 12×12 $\chi_1\,\chi_2\,\chi_4$



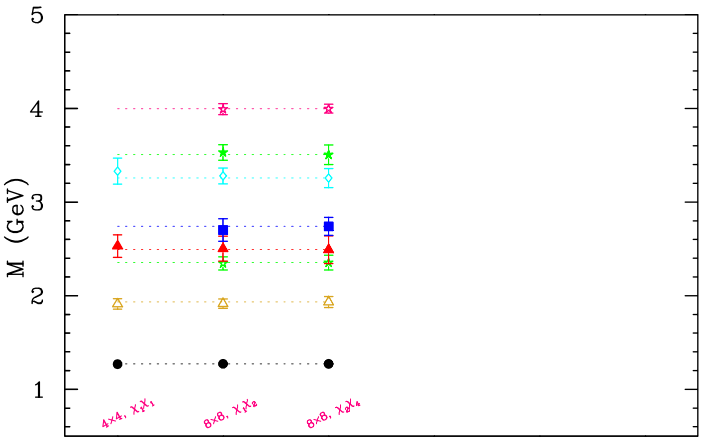
N^+ Spectrum: P-wave $N\,\pi$ threshold



N^+ Spectrum for 2nd heaviest m_q: $4 \times 4 \rightarrow 8 \times 8$ $\chi_1\,\chi_2$



N^+ Spectrum for 2nd heaviest m_q: 8×8 $\chi_2\,\chi_4$



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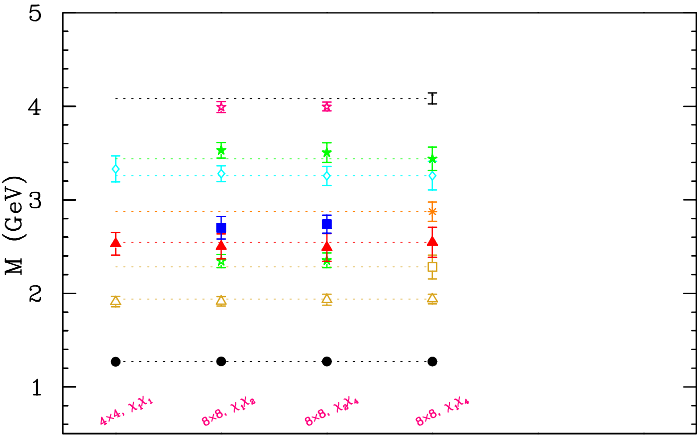

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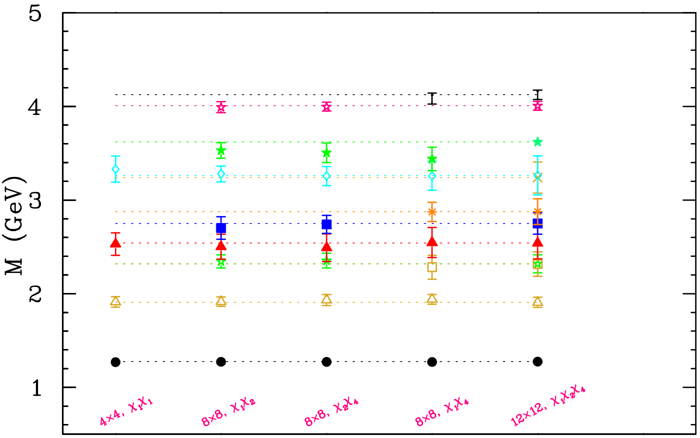
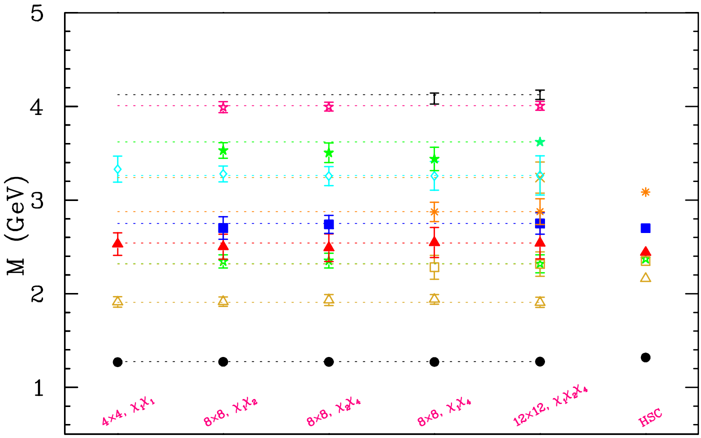
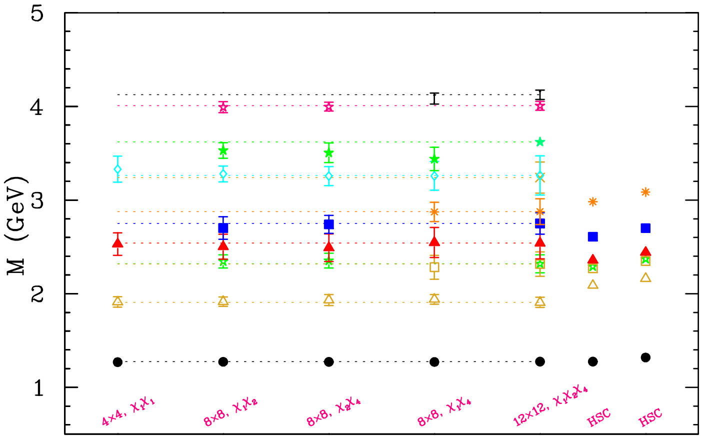


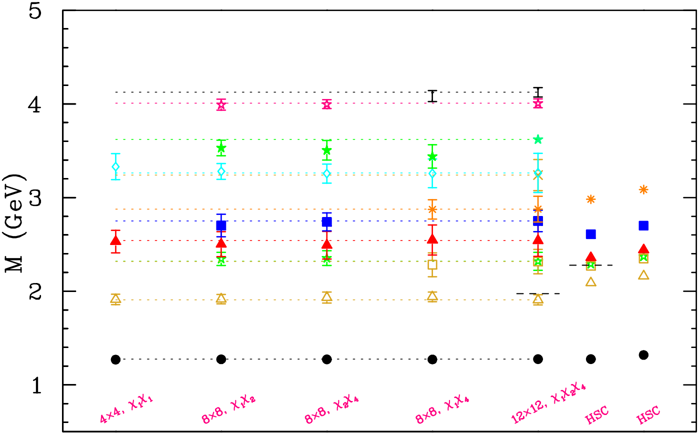
N^+ Spectrum for 2nd heaviest m_q: HSC Comparison



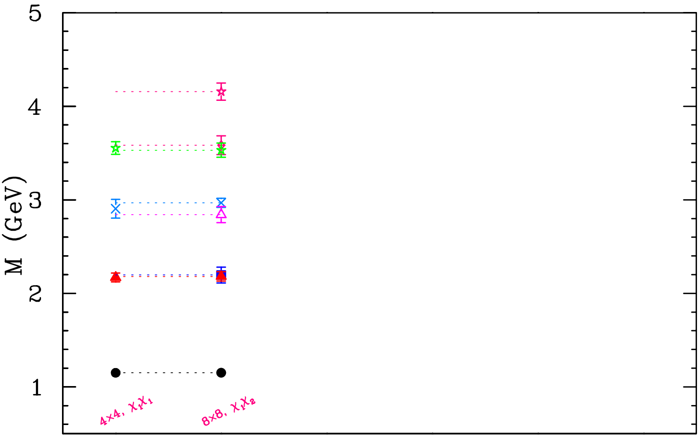
N^+ Spectrum for 2nd heaviest m_q: HSC Rescaled



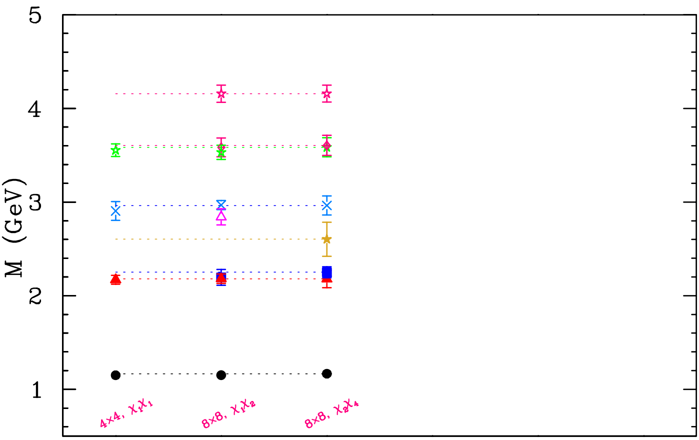
N^+ Spectrum: P-wave $N\,\pi$ thresholds



N^+ Spectrum for 3rd m_q: $4 \times 4 \to 8 \times 8$ $\chi_1 \chi_2$



N^+ Spectrum for 3rd m_q: 8×8 $\chi_2\,\chi_4$



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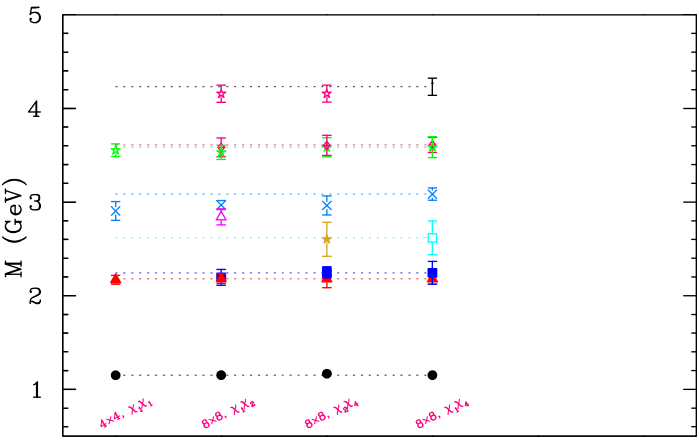
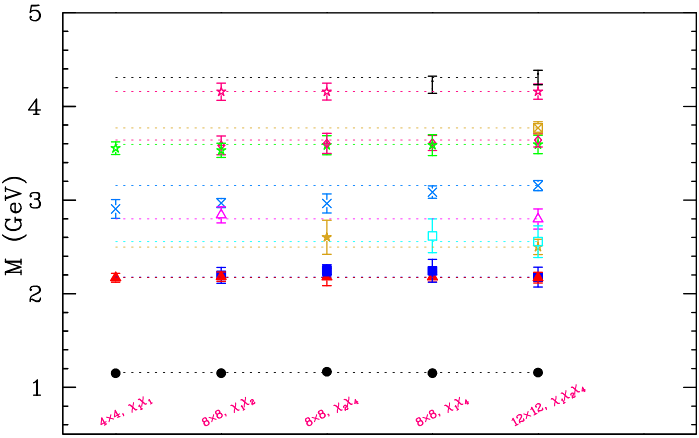
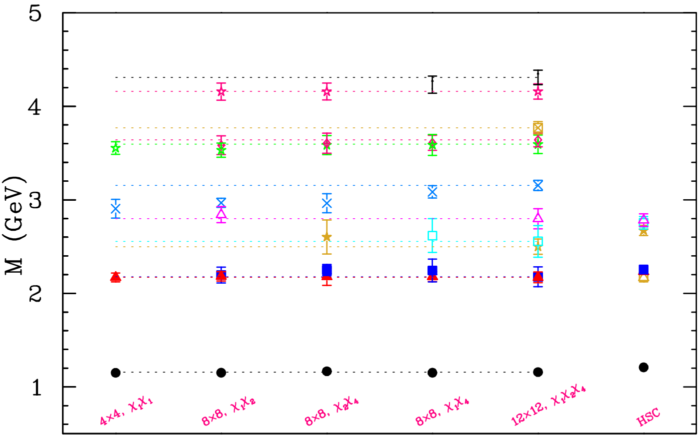


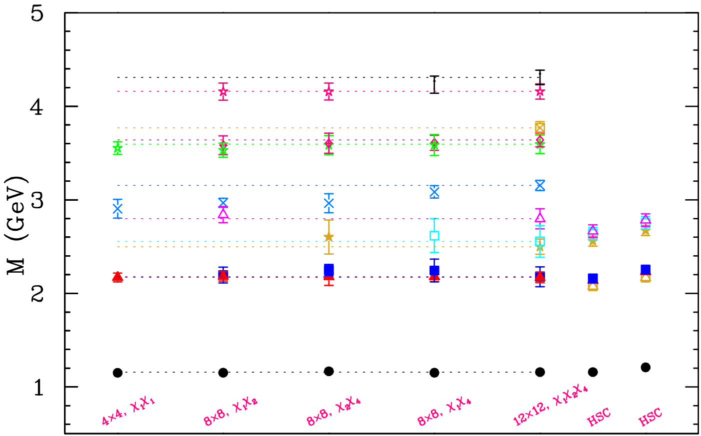
N^+ Spectrum for 3rd m_q: 12×12 $\chi_1 \chi_2 \chi_4$



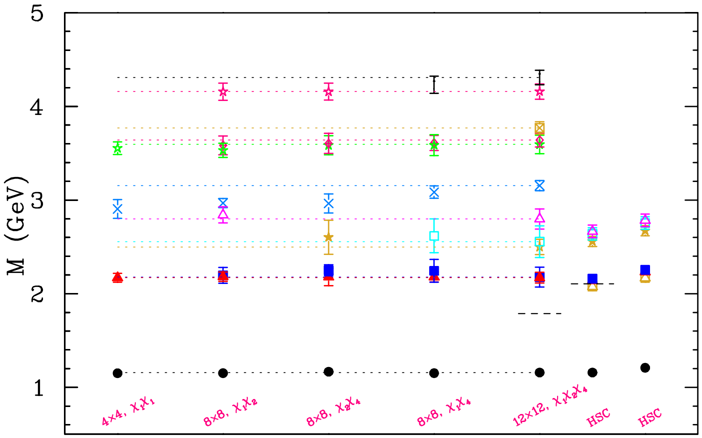
N^+ Spectrum for 3rd m_q: HSC Comparison



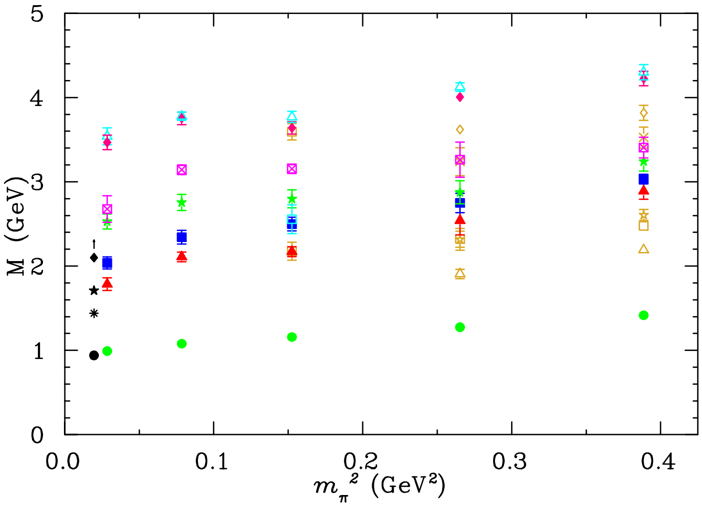
N^+ Spectrum for 3rd m_q: HSC Rescaled



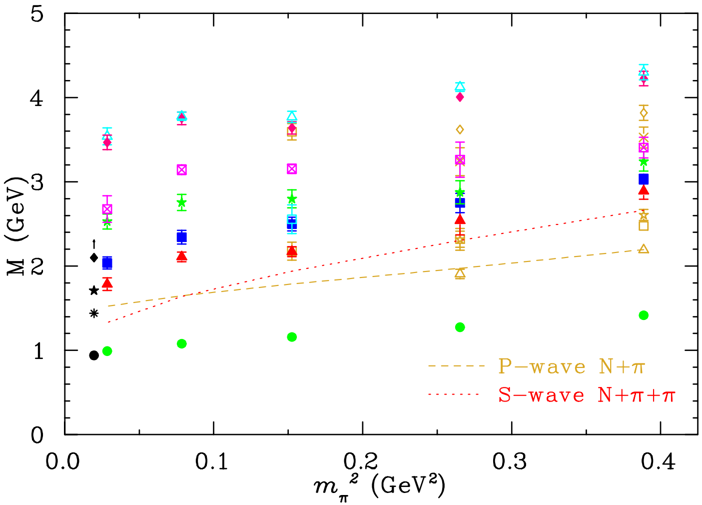
N^+ Spectrum: P-wave $N\,\pi$ thresholds



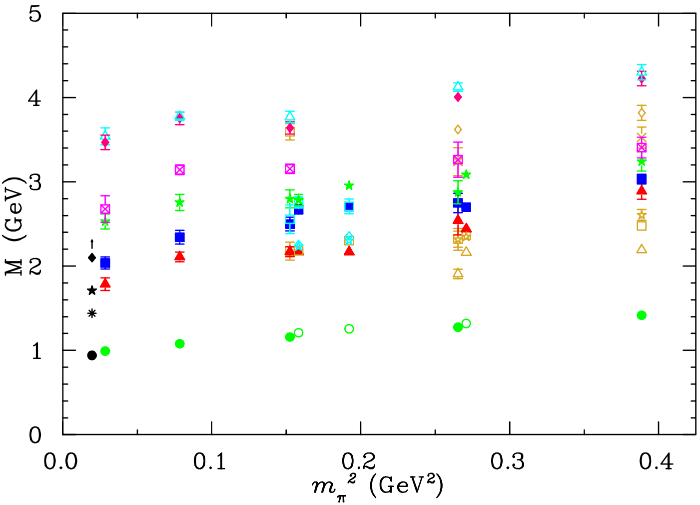
m_π^2 dependence of the N^+ Spectrum



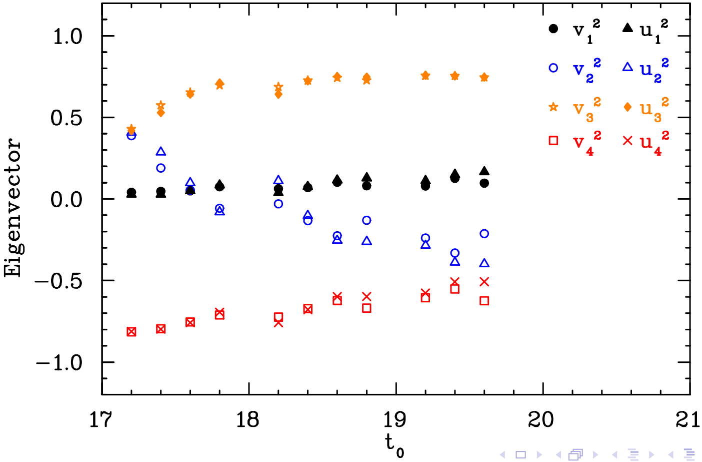
N^+ Spectrum: S and P-wave $N\,\pi$ thresholds



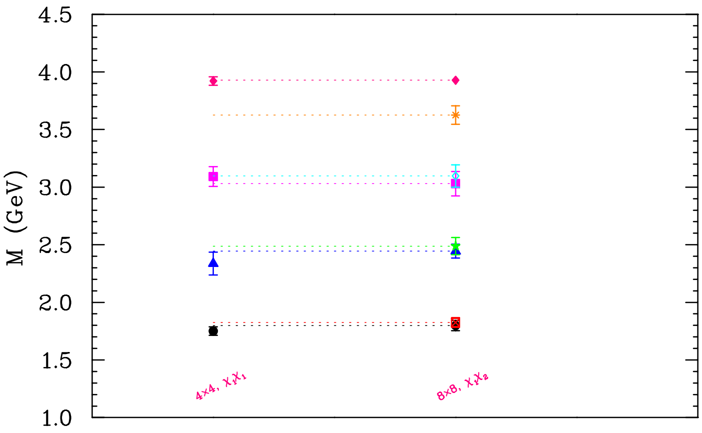
N^+ Spectrum: HSC Comparison



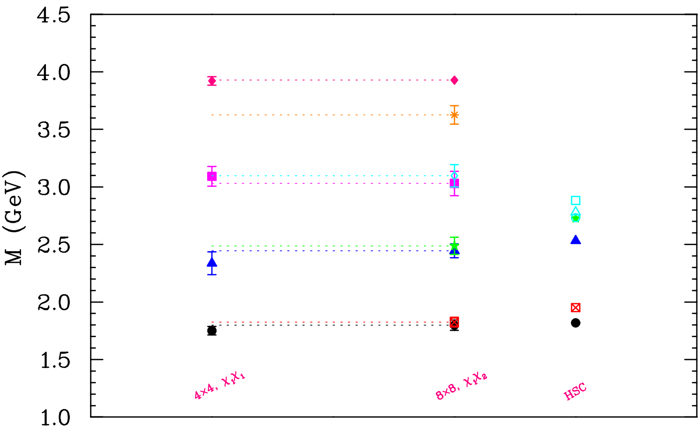
Eigenvectors: Roper state at our lightest mass $m_\pi = 156$ MeV



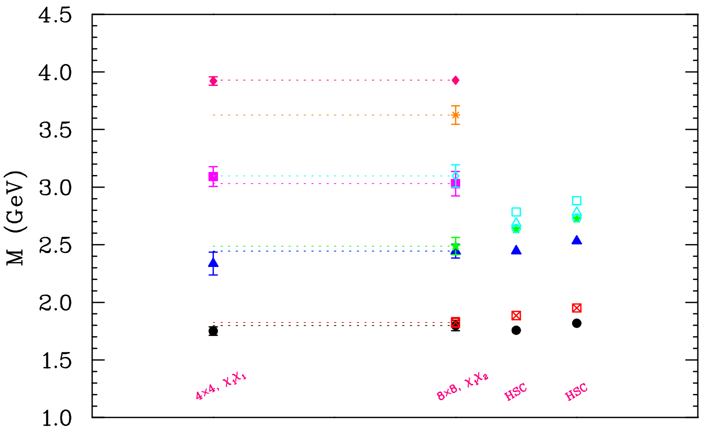
N^- Spectrum for 2nd heaviest m_q: $4 \times 4 \rightarrow 8 \times 8$ $\chi_1\,\chi_2$



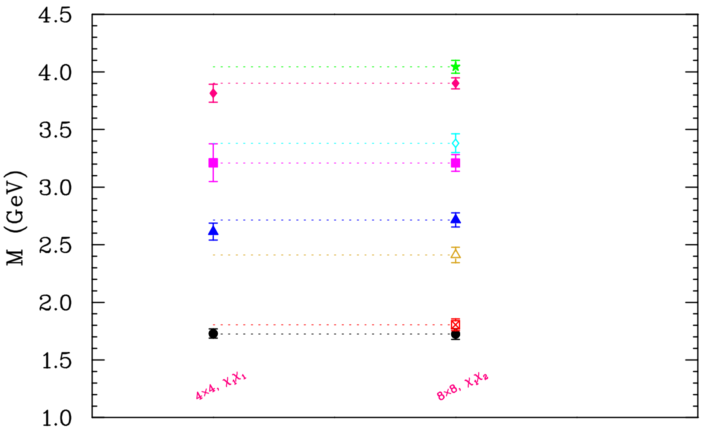
N^- Spectrum for 2nd heaviest m_q: HSC Comparison



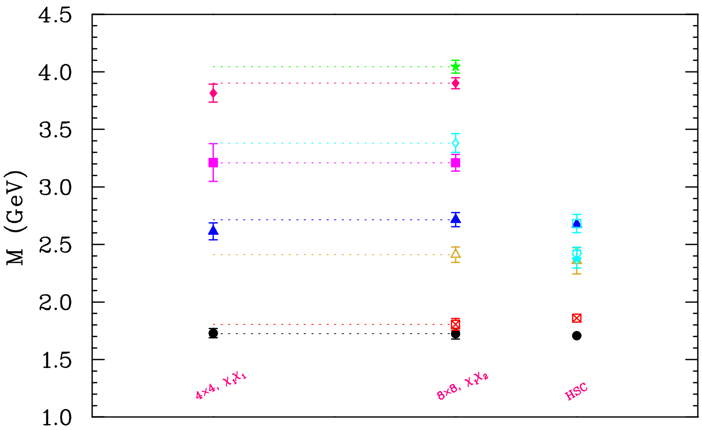
N^- Spectrum for 2nd heaviest m_q: HSC Rescaled



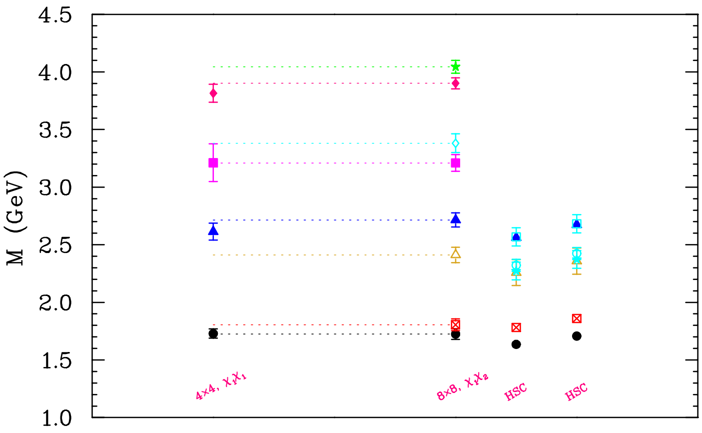
N^- Spectrum for 3rd m_q: $4 \times 4 \rightarrow 8 \times 8$ $\chi_1 \chi_2$



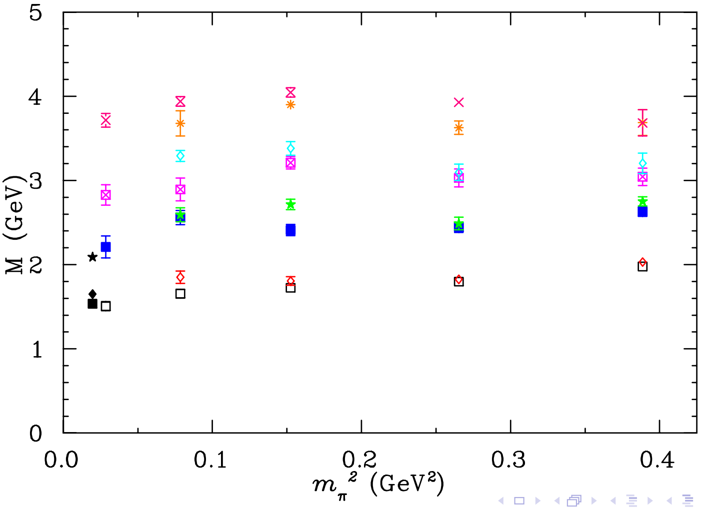
N^- Spectrum for 3rd m_q: HSC Comparison



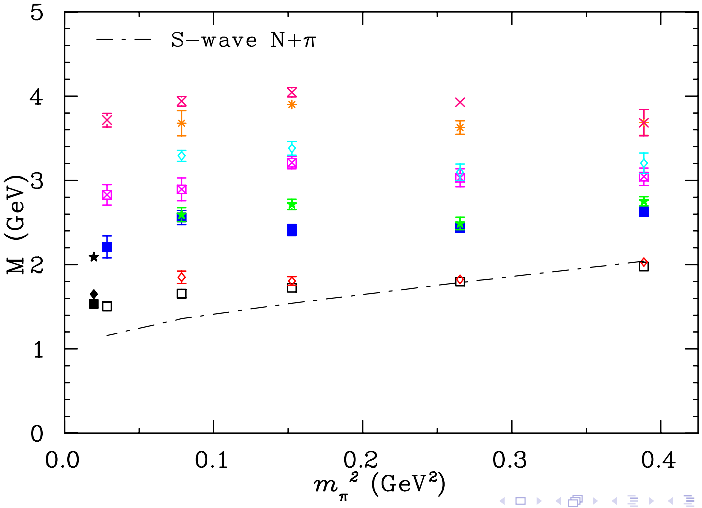
N^- Spectrum for 3rd m_q: HSC Rescaled



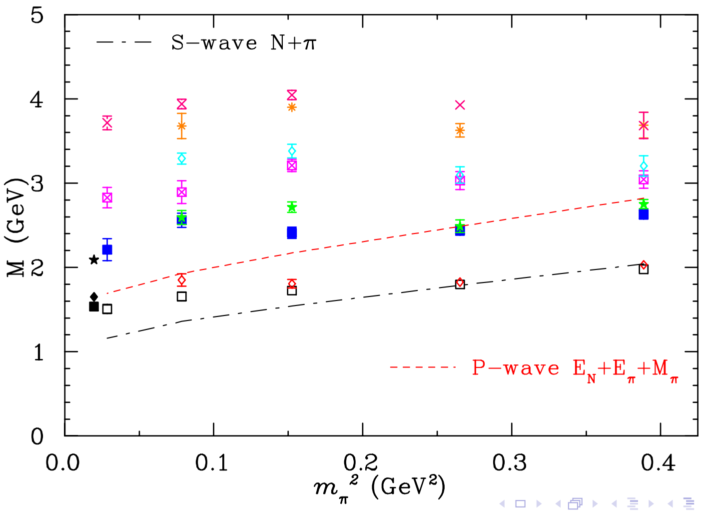
m_π^2 dependence of the N^- Spectrum



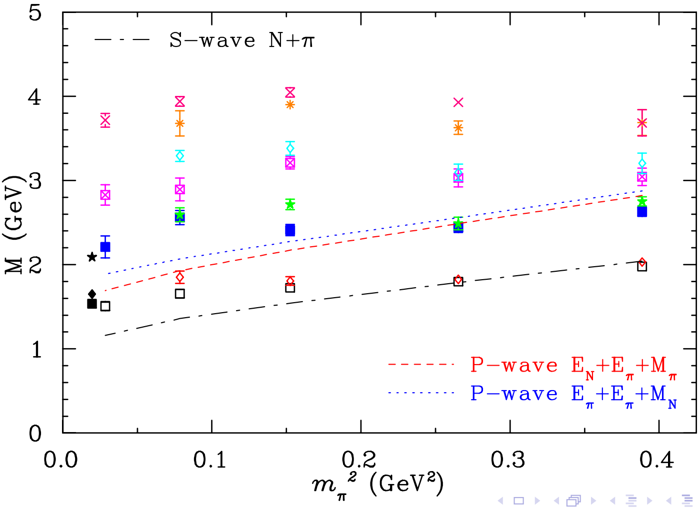
N^- Spectrum: S-wave $N\pi$ threshold



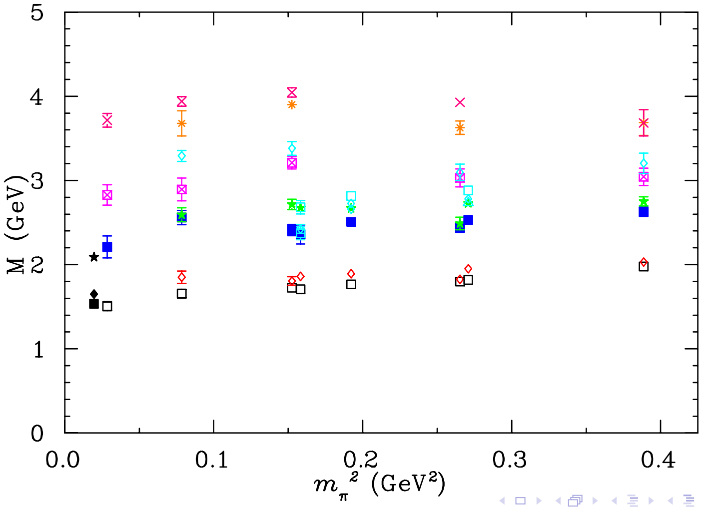
N^- Spectrum: S and P-wave $N\,\pi$ thresholds



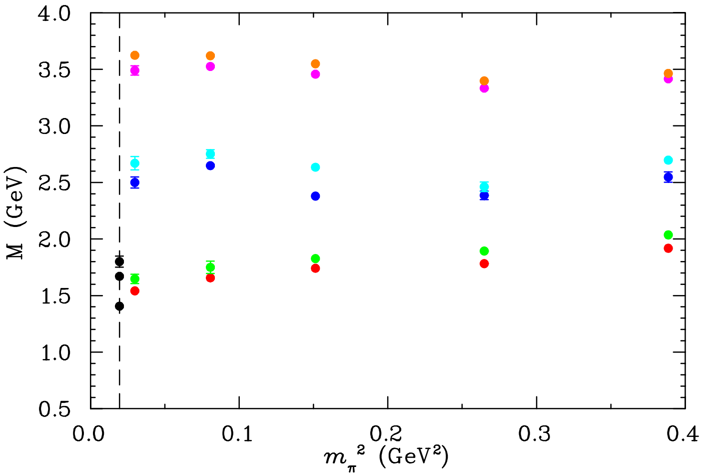
N^- Spectrum: S and P-wave $N\,\pi$ thresholds



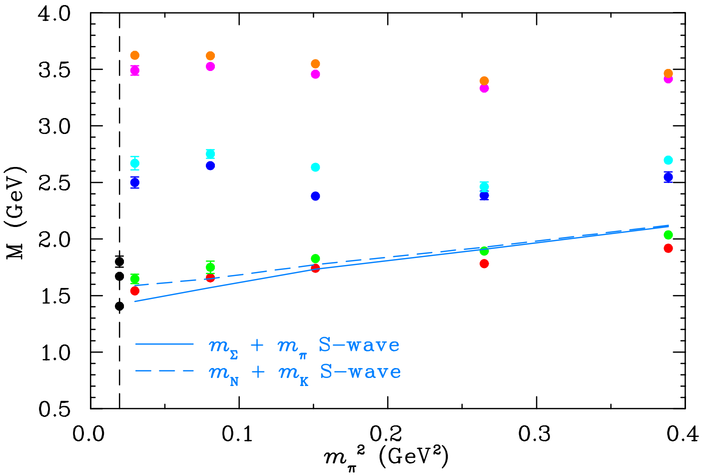
N^- Spectrum: HSC Comparison



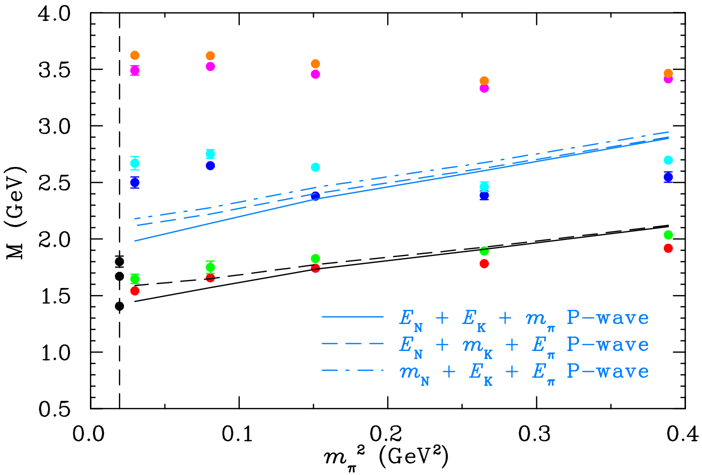
Λ^- Spectrum: 6×6 $\chi_1 \chi_2$



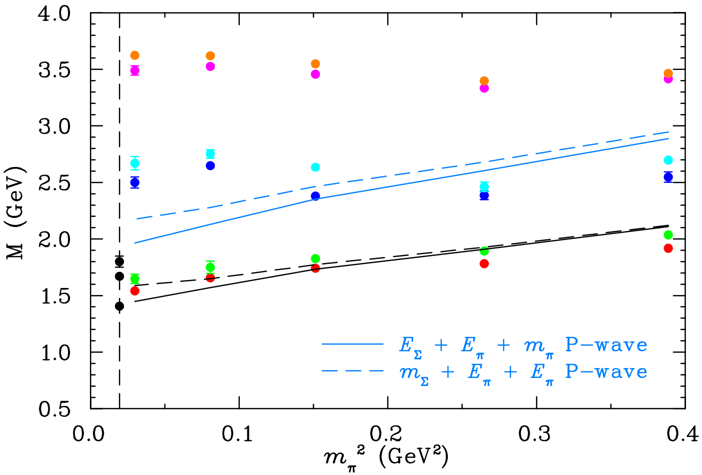
Λ^- Spectrum: S-wave thresholds



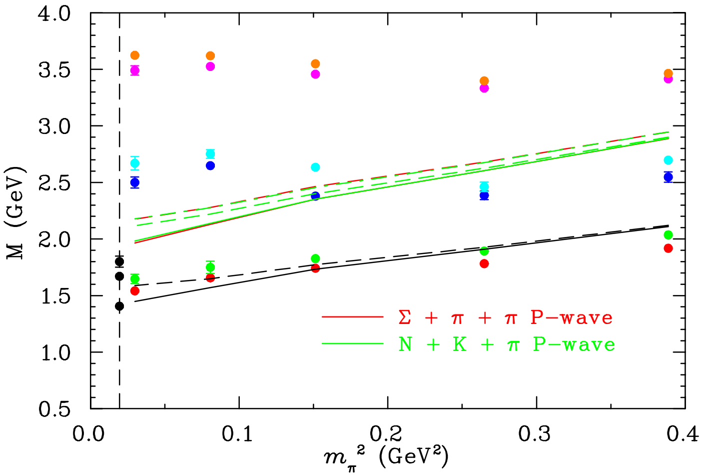
Λ^- Spectrum: P-wave $N\,K\,\pi$ thresholds



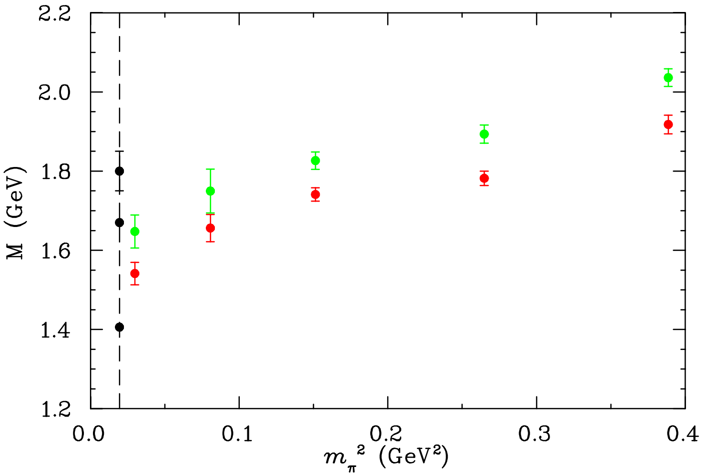
Λ^- Spectrum: P-wave $\Sigma\,\pi\,\pi$ thresholds



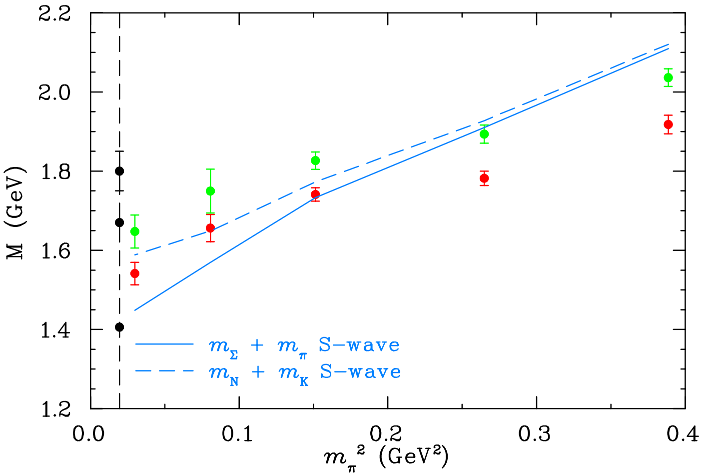
Λ^- Spectrum: All multi-particle thresholds



Low-lying Λ^- spectrum



Low-lying Λ^- spectrum and S-wave thresholds



Summary

- Several fermion-source and -sink smearing levels have been used to construct correlation matrices.
- A variety of 4×4, 8×8 and 12×12 matrices have been considered to explore the eigenstate energies revealed by different interpolating field structures.
- A low-lying Roper state has been identified in both quenched and full QCD using this correlation-matrix based method.
- The approach to the chiral limit is significantly different.
- The two heaviest quark masses considered in the dynamical case provide states consistent with πN multi-particle states.

Summary continued...

- The $N1/2^-$ results in quenched and dynamical QCD reveal significant differences in the approach to the physical point.
- A level crossing between the Roper and $N1/2^-$ states is observed in quenched QCD at $m_\pi \simeq 400$ MeV.
- A level crossing between the Roper and $N1/2^-$ states is anticipated in full QCD at $m_\pi \simeq 150$ MeV, just above the physical pion mass.
- The approach to the experimentally measured masses in full QCD is encouraging.
- The effects of the finite volume on self-energy contributions and associated avoided level crossings remains to be resloved.

Hadron Spectrum Collaboration Results Comparison

- At the heaviest mass compared, we find the same number of N^+ states and qualitative agreement with the spectral energies.
- Finite-volume shifting of the P-wave $N\pi$ threshold is apparent in the spectra.
- Low-lying multi-particle states are suppressed on our large volume lattice for the three lighest quark masses.
- Qualitative agreement of the remaining N^+ states is manifest.
- Qualitative agreement is also observed for the lowest lying N^- states.
- Derivatives provided through the lower components of the Dirac spinors are sufficient to access the $N\frac{1}{2}$ specturm.

$\Lambda(1405)$

- At the lightest mass considered, the Λ^- energy is sitting between the $\Sigma\,\pi$ and $N\,K$ thresholds as the $\Lambda(1405)$ does in Nature.
- While the lowest Λ^- is predominantly flavour-singlet, its energy remains high.
- Again, effects of the finite volume on self-energy contributions and associated avoided level crossings remains to be resolved.
- Future work will expand the study to a complete set of interpolators and explore the flavour mixings of the low-lying states.

Λ^- Spectrum